Robust optimization and probabilistic optimization

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Outline of the talk

- General context and motivation
- Stochastic optimization in short
- Introduction to robust optimization

General context and motivation

Uncertain data: two elements need to be distinguished:

- The set of events possible to come,
- Probabilistic characterization of these events

- Stochastic Optimization:
  - When we know the distribution law characterizing the uncertainty:
    - Multistage (recourse) models,
    - Chance constrains models.

- Robust Optimization
  - When no probabilistic information is available and the uncertainty is characterized by only the set of all events possible to occur.
  - We look for the best solution to be feasible for the set of possible events, which comes to search for optimization in the worst case.

Stochastic optimization in short

- Two methodologies:
  - The approach of optimization on the average
    - Recourse problems
  - The probabilistic approach
    - Chance Constrained Programming

Stochastic optimization in short

Notation:

Ω : set of events ω
A : subset of events (included in Ω)
P : Probabilistic metric on A.
x : decision variables
c : cost function
…

$$\min \quad cx$$
$$A(\omega)x \geq h(\omega)$$
$$x \geq 0$$

$$\begin{array}{ll} \min & cx \\ \text{s.c.} & P\left(A(\omega)x \geq h(\omega)\right) \geq \alpha \\ & x \geq 0 \end{array}$$

Probabilistic optimization

Programming with joint constraints

$$\begin{array}{rl} \min & cx \\ \text{s.c.} & P\left(A(\omega)x \geq h(\omega)\right) \geq \alpha \\ & x \geq 0 \end{array}$$

Programming with separated constraints

$$\begin{array}{rl} \min & cx \\ \text{s.c.} & P\left(A_j(\omega)x \geq h_j(\omega)\right) \geq \alpha_j, \quad \forall j \\ & x \geq 0 \end{array}$$

The case of a finite number K of scenarios

$$\begin{array}{rl} \min & cx \\ \text{s.c.} & P(Ax \geq h) \geq \alpha \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{rll} \min & cx & \\ \text{s.c.} & A_k x \geq \delta_k h_k + (\delta_k - 1).M, & \forall k \in \{1, \ldots, K\} \\ & \sum_{k=1}^{K} \delta_k p_k \geq \alpha & \\ & \delta_k \in \{0, 1\}, & \forall k \in \{1, \ldots, K\} \\ & x \geq 0 & \end{array}$$

Probabilistic optimization

Let's consider:

$$K(\alpha) = \left\{ x \mid P(A(\omega)x \geq h(\omega)) \geq \alpha \right\} \qquad \alpha \in]0,1]$$

1) suppose that A(ω) is a vector, with second term h(ω) uncertain (i.e., A(ω) = A). Then, if F_h is the distribution function of r.v. h, we have:

$$P(Ax \geq h(\omega)) = F_h(Ax)$$

$$K(\alpha) = \{x | F_h(Ax) \geq \alpha\}$$

$$K(\alpha) = \left\{ x | Ax \geq F_h^{-1}(\alpha) \right\}$$

Probabilistic optimization

2) Consider the case with r.v. Gaussian's, (separated constraints).

$$\sum_i A_i x_i \sim N\left(\sum_i \mu_i x_i, \sum_i \sigma_i^2 x_i^2\right)$$

$$\begin{aligned} P\left(\sum_i A_i x_i \geq h\right) \geq \alpha \quad &\Leftrightarrow \quad P\left(\frac{\sum_i A_i x_i - \sum_i \mu_i x_i}{\sqrt{\sum_i \sigma_i^2 x_i^2}} \geq \frac{h - \sum_i \mu_i x_i}{\sqrt{\sum_i \sigma_i^2 x_i^2}}\right) \geq \alpha \\ &\Leftrightarrow \quad P\left(\frac{\sum_i \mu_i x_i - \sum_i A_i x_i}{\sqrt{\sum_i \sigma_i^2 x_i^2}} \leq \frac{\sum_i \mu_i x_i - h}{\sqrt{\sum_i \sigma_i^2 x_i^2}}\right) \geq \alpha \\ &\Leftrightarrow \quad \Phi\left(\frac{\sum_i \mu_i x_i - h}{\sqrt{\sum_i \sigma_i^2 x_i^2}}\right) \geq \alpha \end{aligned}$$

$$K(\alpha) = \left\{ x \mid \sum_i \mu_i x_i - \Phi^{-1}(\alpha) \sqrt{\sum_i \sigma_i^2 x_i^2} \geq h \right\}$$

Probabilistic optimization

- The probabilistic approach is difficult in general to tackle:
  - Only the r.v. Gaussians case can be handled, the others are difficult, including the uniform case.
  - The set of feasible solutions is not necessary convex.

- General context and motivation
- Stochastic optimization in short
- **Introduction to robust optimization**

Introduction to robust optimization

- Robust optimization is used when no probabilistic information is available; the only information is the set of possible events that can occur.
- We look for the best solution to be feasible for the set of possible events, which comes to search for optimization in the worst case.
  - Usual optimization criterion studied: *Minmax cost*

Introduction to robust optimization

- *minmax cost* :

"minmax cost", (Soyster, 1970): Optimizing for the worst scenario: the robust solution should, for instance, minimize the maximal cost resulted for the considered scenarios (a finite number K).

$$
\begin{array}{lllll}
\min & \max_k c_k x & = & \min & z \\
\text{s.c.} & A_k x \geq b_k, \forall k \in \{1,\ldots,K\} & & \text{s.c.} & c_k x \leq z, \forall k \in \{1,\ldots,K\} \\
& x \in X & & & A_k x \geq b_k, \forall k \in \{1,\ldots,K\} \\
& & & & x \in X
\end{array}
$$

- The size of the robust programming increases with the number of scenarios.
- The complexity can radically change when we consider several scenarios:
  - When the weights of links in a graph are uncertain, the shortest path problem becomes NP-difficult, even for two scenarios (|K| = 2).

Introduction to robust optimization

– Some questions may be raised:

1. Which is the best compromise between the feasibility and the robustness ?

2. How one can build the “best” set of events for which the solution should be robust ?

3. Is there any methodology for “optimization under uncertainty” with moderated theoretical and practical complexity?

State of art

Some works on the link between robust optimization and the feasibility probability of the solutions:

- A. Ben-Tal and A. Nemirovski, Robust solutions of Linear Programming problems contaminated with uncertain data, Math. Prog. (2000)
- [BS04] D. Bertsimas et M. Sim, The Price of Robustness, Operations Research (2004)
- A. Nemirovski and A. Shapiro, Convex Approximations of Chance Constrained Programs, à paraître dans SIAM J. Optim.

- ... and others!

Extended robust optimization, Bertsimas and Sim, [BS04]

The approach proposed by Bertsimas and Sim can be written as below:

$$(CCP)\begin{cases} \min & cx \\ \text{s.c} & \Pr(Ax \le b) \ge 1-\varepsilon \\ & x \in \mathbb{N} \end{cases}$$

$$\downarrow$$

$$(RP)\begin{cases} \min & cx \\ \text{s.c} & A'x \le b' \\ & x \in \mathbb{N} \end{cases}$$

with A’, b’ such that: $\forall x \in \mathbb{N}$ $A'x \le b' \Rightarrow \Pr(Ax \le b) \ge 1 - \varepsilon$.
Thus, any feasible solution for (RP) is also feasible for (CCP)

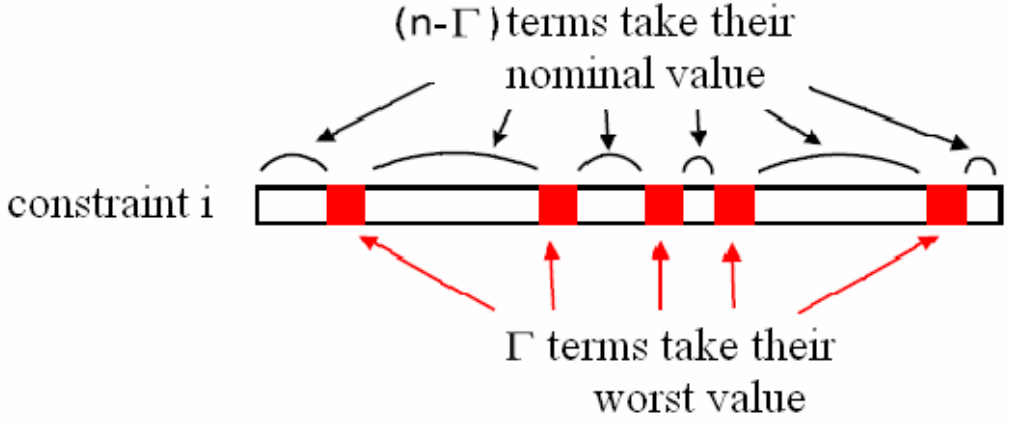
This model is directly usable for ILP problems.

Model description ([BS04])

Notation : I gives the set of constraints, J gives the set of columns

Hypotheses :

- Each term A_{ij} is uncertain: $A_{ij} \in [\overline{A}_{ij} - \hat{A}_{ij}, \overline{A}_{ij} + \hat{A}_{ij}]$.
- $\Gamma_i \in \{0, \ldots, |J|\}$: at most Γ_i can take their worst value



$$\text{constraint } i \ : \ \max_{S \subseteq J : |S| = \Gamma_i} \sum_{j \in S} (\overline{A}_{ij} + \hat{A}_{ij}) x_j + \sum_{j \in J \setminus S} \overline{A}_{ij} x_j \leq b_i$$

Model description ([BS04])

The robust problem (initially)

$$\begin{aligned} &\min && cx \\ &\text{s.c.} && \sum_{j\in S}(\overline{A}_{ij} + \hat{A}_{ij})x_j + \sum_{j\in J\setminus S}\overline{A}_{ij}x_j \le b_i \quad \forall S \subseteq J \text{ t.q. } |S| = \Gamma_j, \ \forall\, i \in I, \\ &&& x_j \ge 0 \end{aligned}$$

The robust problem proposed by [BS04]

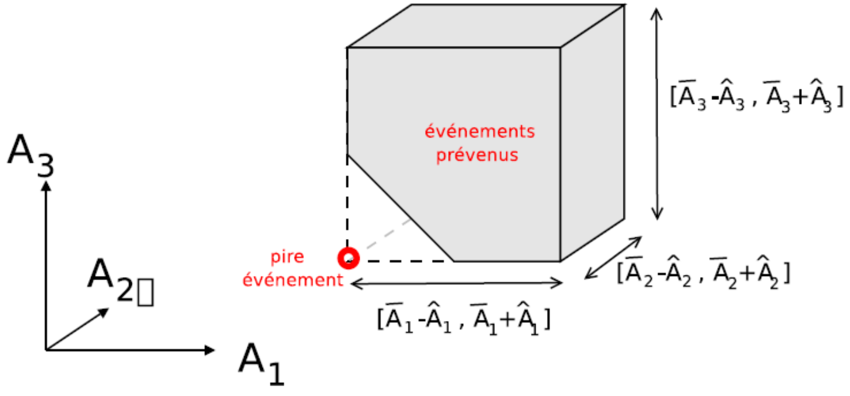
$$\begin{aligned} &\min && cx \\ &\text{s.c.} && \sum_{j\in J}\overline{A}_{ij}x_j + z_i\Gamma_i + \sum_{j\in J} p_{ij} \le b_i, && \forall\, i \in I, \\ &&& z_i + p_{ij} \le \delta_{ij}x_j, && \forall (i,j) \in I \times J, \\ &&& x_j \ge 0,\ p_{ij} \ge 0,\ z_i \ge 0, && \forall (i,j) \in I \times J. \end{aligned}$$

The robust variant of an LP is also an LP

The robust variant of an ILP is also an ILP.

Why the model is interesting?

- The model preserves the linearity;
- The robust variant can be given through a compact formulation, comparable to the nominal problem.
- "reasonable modeling", it avoids the worst possible matrix (or events), in general less probable:



Feasibility probability $1 - \varepsilon \rightarrow$ choice of coefficient Γ_i

Thank you